Review: Complex Numbers

CS 450: Introduction to Digital Signal and Image Processing

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Complex Numbers

A complex number is one of the form

$$a + bi$$

where

$$i = \sqrt{-1}$$

a: *real part*

b: *imaginary part*

Complex Arithmetic

- When you add two complex numbers, the real parts and imaginary parts add independently:

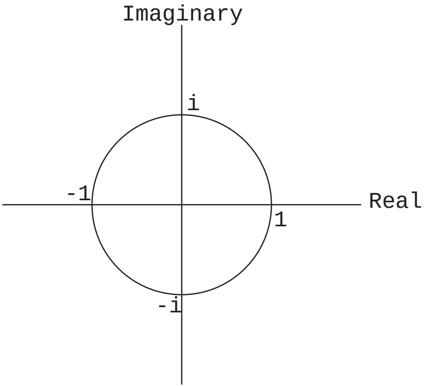
$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- When you multiply two complex numbers, you cross-multiply them like you would polynomials:

$$\begin{aligned}(a + bi) * (c + di) &= ac + a(di) + (bi)c + (bi)(di) \\ &= ac + (ad + bc)i + (bd)(i^2) \\ &= ac + (ad + bc)i - bd \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

The Complex Plane

Complex numbers can be thought of as points in the complex plane:



Magnitude and Phase

- The length is called the *magnitude*:

$$|a+bi| = \sqrt{a^2+b^2}$$

- The angle from the real-number axis is called the *phase*:

$$\phi(a+bi) = \tan^{-1}\left(\frac{b}{a}\right)$$

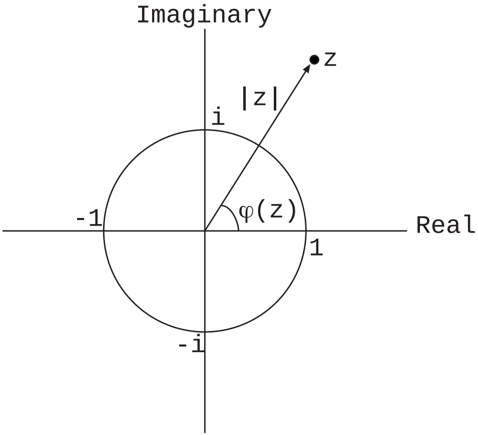
- When you multiply two complex numbers, their magnitudes multiply:

$$|xy| = |x||y|$$

  and their phases add:

$$\phi(xy) = \phi(x) + \phi(y)$$

Magnitude and Phase in the Complex Plane



Complex Conjugates

- Complex number z:

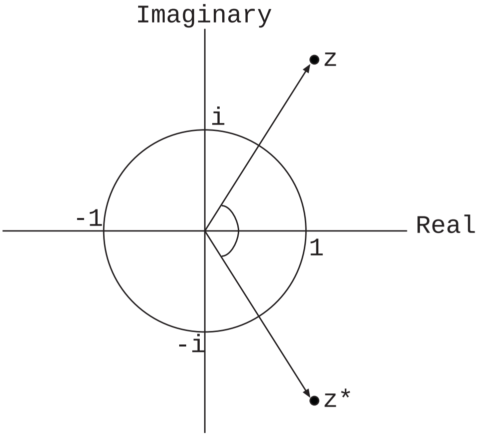
$$z = a + bi$$

- Its *complex conjugate*:

$$z^* = a - bi$$

- The complex conjugate z^* has
  - the same real part but opposite imaginary part, and
  - the same magnitude but opposite phase.

Complex Conjugates in the Complex Plane



Complex Conjugates

- Adding $z + z^*$, cancels the imaginary parts to leave a real number:
$$(a + bi) + (a - bi) = 2a$$
- Multiplying $z * z^*$ gives the real number equal to $|z|^2$:
$$\begin{aligned} (a + bi)(a - bi) &= a^2 - (bi)^2 \\ &= a^2 + b^2 \end{aligned}$$

Linear Algebra with Complex Numbers

- The inner product of two complex-valued vectors involves multiplying each component of one of the vector not by the other but by the *complex conjugate* of the other:

$$\bar{u} \cdot \bar{v} = \sum_k \bar{u}[k] \ \ \bar{v}[k]^*$$

- The length of a complex-valued vector is thus a real number:

$$\|\bar{u}\|^2 = \bar{u} \cdot \bar{u} = \sum_k \bar{u}[k] \ \ \bar{u}[k]^*$$

Magnitudes and Phases - revisited

- Remember that under complex multiplication
  - magnitudes multiply
  - phases add
- We can do the same thing using exponents:

$$(a_1 e^{b_1})(a_2 e^{b_2}) = a_1 a_2 e^{(b_1+b_2)}$$

- *Let's encode complex numbers using exponential notation to make it easier to work with magnitude and phase*

Euler's Formula

- *Euler's formula* uses exponential notation to encode complex numbers—uses i in the exponent to differentiate from real numbers
- Euler's formula (definition):

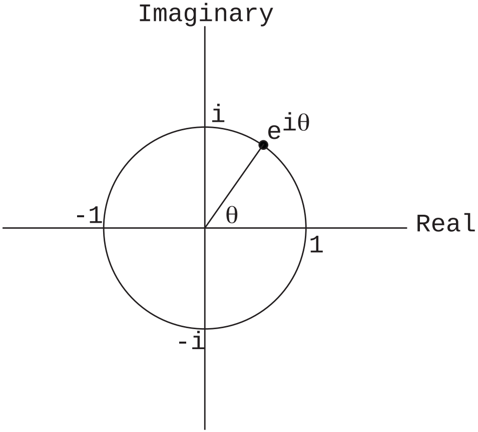
$$e^{i\theta} = \cos\theta + i\sin\theta$$

  $e^{i\theta}$ is the vector with magnitude 1.0 and phase θ
- Any complex number z can be written as

$$z = |z|\, e^{i\phi(z)}$$

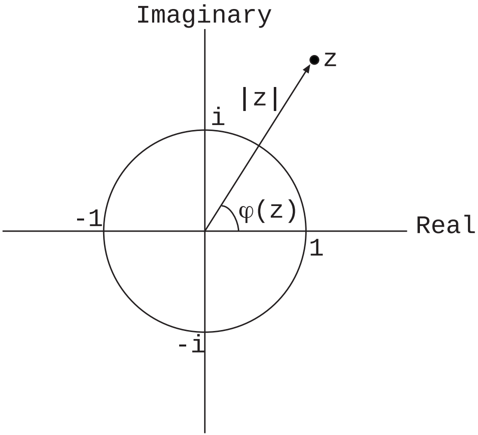
Euler's Formula: Graphical Interpretation

$e^{i\theta}$



Euler's Formula: Graphical Interpretation

$$z = |z|e^{i\phi(z)}$$



Euler's Formula: Application

What is $(2+2i)(-3+3i)$?

Suppose that we already have these numbers in magnitude-phase notation:

$$
\begin{array}{cc}
|2+2i| = 2\sqrt{2} & |-3+3i| = 3\sqrt{2} \\
\phi(2+2i) = \frac{\pi}{4} & \phi(-3+3i) = \frac{3\pi}{4} \\
2+2i = 2\sqrt{2}\ e^{i\pi/4} & -3+3i = 3\sqrt{2}\ e^{i3\pi/4}
\end{array}
$$

$$
\begin{aligned}
(2+2i)(-3+3i) &= \left(2\sqrt{2}\ e^{i\pi/4}\right)\left(3\sqrt{2}\ e^{i3\pi/4}\right) \\
&= 12\ \ e^{i\pi} \\
&= -12
\end{aligned}
$$

Powers of Complex Numbers

Suppose that we take a complex number

$$z = |z| \; e^{i\phi(z)}$$

and raise it to to some power n:

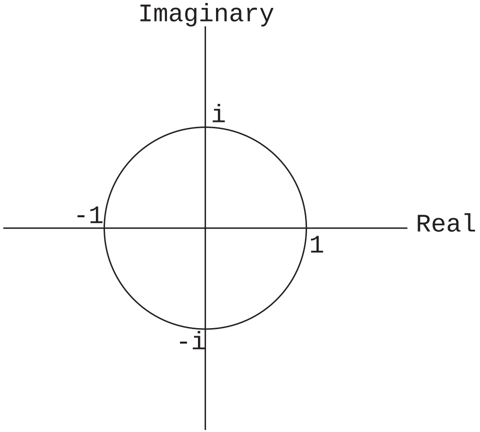
$$\begin{aligned} z^n &= \left[|z| \; e^{i\phi(z)}\right]^n \\ &= |z|^n \; e^{in\phi(z)} \end{aligned}$$

z^n has magnitude $|z|^n$ and phase $n\left[\phi\left(z\right)\right]$.

Powers of Complex Numbers: Example

What is i^n for various n?

$$i = e^{i\pi/2}$$
$$i^0 = e^{i0} = 1$$
$$i^1 = e^{i\pi/2} = i$$
$$i^2 = e^{i2\pi/2} = -1$$
$$i^3 = e^{i3\pi/2} = -i$$
$$i^4 = e^{i4\pi/2} = 1$$
$$\vdots$$



Powers of Complex Numbers: Example

What is $\left(e^{i\pi/4}\right)^n$ for various n?

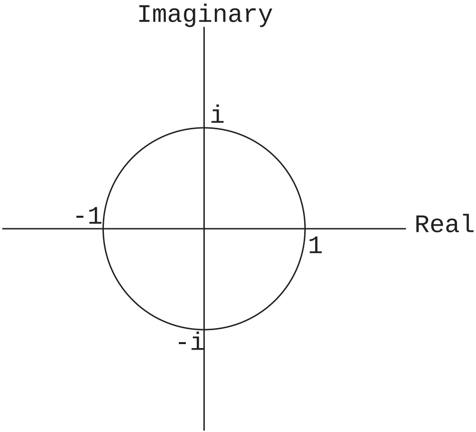
$\left(e^{i\pi/4}\right)^0$

$\left(e^{i\pi/4}\right)^1$

$\left(e^{i\pi/4}\right)^2$

$\left(e^{i\pi/4}\right)^3$

$\left(e^{i\pi/4}\right)^4$



Summary: Complex Numbers

- Can represent in (real,imaginary) Cartesian form
- Can represent in (magnitude,phase) polar form
- Magnitude = distance from 0 (same idea as absolute value)
- Phase = angle with the real axis
- Euler's theorem: exponential notation for (magnitude,phase)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z = |z|\ e^{i\phi(z)}$$

- Complex conjugate: $z^* = a - bi = |z|\ e^{-i\phi(z)}$
- Raising a complex number to a power:
  z^n has magnitude $|z|^n$ and phase $n\left[\phi\left(z\right)\right]$